Ruled Surfaces

Note Title 5/20/2020

Definition: A surface X is <u>ruled</u> if it is birational to $C \times \mathbb{P}^1$, C: smooth curve.

ex. ① $C \times \mathbb{P}^1$

② $X = \mathbb{P}(E)$, $\begin{matrix} E \\ \downarrow \\ C \end{matrix}$ rank 2 vector bundle.

temporary auxiliary

Definition: A surface X is <u>geometrically ruled</u> if $\exists$ $X \xrightarrow{p} C$ smooth morphism, $p^{-1}(c) \cong \mathbb{P}^1$, $\forall c \in C$

C: smooth curve

Theorem 1. If $X \xrightarrow{p} C$ is geometrically ruled, then $\exists U \subseteq C$ s.t $p^{-1}(U) \cong U \times \mathbb{P}^1$

Enrique – Noether

Zariski open

In particular, geometrically ruled $\Longrightarrow$ ruled

pf: Step 1. $H^2(X, \mathcal{O}_X) = 0$

Otherwise, $0 \neq H^2(X, \mathcal{O}_X) \cong H^0(X, K_X)$

Serre duality

ie. $\exists$ effective anti-canonical divisor K_X

F: generic fibre

$\mathcal{O}_{\mathbb{P}^1}(-2) \cong K_F \cong F + K_X\big|_F$

deg $\downarrow$ adjuntion formula $\searrow$ deg

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$F.F + K_X.F$

F: fibre $\quad \overset{\|}{0} \quad \overset{\text{IV}}{0}$ $\quad K_X$ effective $\quad \mathcal{O}(F)$ no base point

Step 2. $\exists$ divisor S s.t $S.F = 1$. Later S can be realized as a section.

$$H^1(X, \mathcal{O}_X^*) \overset{c_1}{\twoheadrightarrow} H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathcal{O}_X) = 0$$

It suffices to prove that $\exists\ s \in H^2(X, \mathbb{Z})$ s.t $s.f = 1$, $\quad f = [F]$

Poincare duality: $H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \longrightarrow \mathbb{Z}$ non-degenerate

$$\Rightarrow 0 \to \mathrm{Tor} \to H^2(X, \mathbb{Z}) \twoheadrightarrow \mathrm{Hom}(H^2(X, \mathbb{Z}), \mathbb{Z})$$

$K_X.f = -2 \quad \therefore f \notin \mathrm{Tor}$

$$\begin{array}{c} \psi \\ f \end{array} \longmapsto (a \mapsto a.f)$$

$(a.f) \triangleleft \mathbb{Z}$ ideal

$\| \quad$ $d = 1$ or 2 in this case

$d\mathbb{Z}$

Poincare duality $\Rightarrow \exists f' \in H^2(X, \mathbb{Z})$ s.t $a.f' = \frac{1}{d} a.f$, $\forall a \in H^2(X, \mathbb{Z})$

$\therefore f = df'$ modulo torsion

Observation: $a^2 + a.K_X \in (2) \triangleleft \mathbb{Z}$, $\forall a \in H^2(X, \mathbb{Z})$

$\Rightarrow \quad d = 1$. In particular, $\exists\, s \in H^2(X, \mathbb{Z})$

$s \cdot f = 1$

Step 3. $\quad 0 \to \mathcal{O}_X(S+(n-1)F) \to \mathcal{O}_X(S+nF) \to \mathcal{O}_F(1) \to 0$

$$H^0(X, \mathcal{O}_X(S+nF)) \xrightarrow[\twoheadrightarrow]{n \gg 0} H^0(F, \mathcal{O}_F(1)) \to H^1(X, \mathcal{O}_X(S+(n-1)F)) \xrightarrow[\cong]{n\gg 0} H^1(X, \mathcal{O}_X(S+nF)) \to 0$$

$H^0(X, \mathcal{O}_X(S+nF)) \supseteq V$, $\exists\ V$ pencil; $H^0(F, \mathcal{O}_F(1)) \cong \mathbb{C}^2$, no base point

$h^1(X, \mathcal{O}_X(S+nF))$ stabilized as $n \gg 0$

$H^1(X, \mathcal{O}_X(S+nF)) = H^1(F, \mathcal{O}_F(1))$

Base locus of V is contained in fibres $F_{c_1}, \cdots, F_{c_k}$ other than F.

disjont from F. thus can't have a component.

$U = C \setminus \{c_1, \cdots, c_k\}$, then $V|_U$ is base point free

$$\Rightarrow \quad p^{-1}(U) \xrightarrow{|V|_U|} \mathbb{P}^1 \quad \text{isomorphism on fibres}$$

$$\downarrow$$

$$U$$

Thus, $p^{-1}(U) \cong U \times \mathbb{P}^1$

injective, surjective, differential $\neq 0$

Proposition 1: $X \xrightarrow{p} C$ geometrically ruled surface

then $\exists E$ rank 2 vector bundle on C

s.t.

$$\begin{tikzcd} X \arrow[r, "\cong"] \arrow[dr, "p"'] & \mathbb{P}_C(E) \arrow[d] \arrow[dl, phantom, "\circlearrowleft"] \\ & C \end{tikzcd}$$

always projective

In particular, the quotient sub-bundles of E is equivalent to sections of p

pf: Notice that in the Step 3 of the proof of Theorem 1, one can run the argument again w/ $F = F_{c_i}$

$\rightsquigarrow X \longrightarrow C$ is locally trivial.

& classified by $H^1(C, PGL(2, \mathcal{O}_C))$

$$1 \to \mathbb{C}^* \to GL(2,\mathbb{C}) \longrightarrow PGL(2,\mathbb{C}) \longrightarrow 1$$

global version

$$1 \to \mathcal{O}_C^* \to GL(2, \mathcal{O}_C) \to PGL(2, \mathcal{O}_C) \to 1$$

$$H^1(C, \mathcal{O}_C^*) \longrightarrow H^1(C, GL(2, \mathcal{O}_C)) \longrightarrow H^1(C, PGL(2, \mathcal{O}_C)) \to H^2(C, \mathcal{O}_C^*) = 0$$

Grothendieck

Thus all geometrically ruled surfaces are of the form $\mathbb{P}_C(E)$

$\mathbb{P}_C(E) \cong \mathbb{P}_C(E')$ iff $E \cong E' \otimes \mathcal{L}$, $\mathcal{L} \in H^1(C, \mathcal{O}_C^*)$

ex. If $C \cong \mathbb{P}^1$, then $E \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$

Grothendieck

$$\mathbb{P}_C(E) \cong \mathbb{P}_C(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)) \cong \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a-b)) = \mathbb{F}_{|a-b|}$$

All geometric ruled surfaces over $\mathbb{P}^1$ are Hirzebruch surfaces $\mathbb{F}_{n \in \mathbb{Z}}$

Thus, we want to deviate slightly to understand rank 2 bundles on curves.

Lemma 1: E = rank 2 vector bundle on C

① $\exists$ line bundles L_1, L_2 s.t $0 \longrightarrow L_1 \longrightarrow E \longrightarrow L_2 \longrightarrow 0$

② If $h^0(E) \geq 1$, then can take $L_1 \cong \mathcal{O}_C(D)$, $D \geq 0$.

(③ If $h^0(E) \geq 2$, $\deg E > 0$, can assume $D > 0$)

pf: Notice that every proper curve is projective.

By tensoring high power of ample line bundle, we may assume that $\exists \underset{\neq 0}{s} \in H^0(E)$.

by the theorem of Serre

$$\begin{tikzcd} \check{E} \arrow[r, ".s"] \arrow[dr] & \mathcal{O}_C \\ & \mathcal{O}_C(-D) \arrow[u, hook] \end{tikzcd} \quad \check{L}_1 = \mathcal{O}_C(-D), \quad D>0$$

Kernel is also an invertible sheaf.

ideal sheaf are invertible on curves

Taking dual implies ① & ②

③ Notice that $D = \{ \underset{\neq 0}{s} = 0 \}$ above

$h^0(E) \geq 2$ i.e. $\exists \underset{\neq 0}{s_1, s_2} \in H^0(E)$

$\Longrightarrow \underset{\neq 0}{s_1 \wedge s_2} \in H^0(\overset{2}{\wedge} E)$ has a zero p

$\deg \overset{2}{\wedge} E > 0$

i.e. $\underbrace{\lambda_1 s_1 + \lambda_2 s_2}_{\in H^0(E)}(p) = 0$ for some λ_1, λ_2

In particular, any rank 2 bundle E on C is an extension of line bundles of L_1, L_2

parametrized by $\mathrm{Ext}^1(L_1, L_2)$
$= \mathrm{Ext}^1(\mathcal{O}_X, L_1^{-1} \otimes L_2) = H^1(L_1^{-1} \otimes L_2)$

Indeed $\quad 0 \to L_1 \otimes L_2^{-1} \to E \otimes L_2^{-1} \to \mathcal{O}_C \to 0$

$$E = L_1 \oplus L_2 \iff \begin{array}{ccccc} H^0(E \otimes L_2^{-1}) & \longrightarrow & H^0(\mathcal{O}_C) & \xrightarrow{\partial} & H^1(L_1 \otimes L_2^{-1}) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ s & \longmapsto & 1 & \longmapsto & \partial(1) = 0 \end{array}$$

i.e $\quad \partial(1) \in H^1(L_1 \otimes L_2^{-1})$ is the obstruction for splitting.

ex. (elliptic curve) E indecomposable

$\underline{E \cong E' \otimes L}$, $L \in \mathrm{Pic}(C)$

twist s.t $\deg E' = 0, 1$ $\quad E'$ = non-trivial extension of $\mathcal{O}_C$ by $\mathcal{O}_C$

or $\quad$ = $\quad \mathcal{O}_C(p)$ by $\mathcal{O}_C$

ex. (genus $g \geq 2$ curve)

$$\begin{array}{ccc} E_s & \hookrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ s & \in & S \end{array}$$

$E_s \ncong E_{s'}$ if $s \neq s'$

$\overset{2}{\wedge} E_s$ constant

E_s indecomposable

$\dim S = 3g - 3$

Theorem 2. C = smooth curve, not $\mathbb{P}^1$

Then minimal models of $C \times \mathbb{P}^1$ is $\mathbb{P}_C(V)$ for some V, $\mathrm{rk}(V) = 2$

Topology of geometrically ruled surface

Proposition 2: E = rank 2 bundle / C $\quad X = \mathbb{P}_C(E) \xrightarrow{p} C$

① $\mathrm{Pic}(X) \cong p^*\mathrm{Pic}(C) \oplus \mathbb{Z}h$, $\quad h = [\mathcal{O}(1)]$

② $H^2(X, \mathbb{Z}) \cong \mathbb{Z}f \oplus \mathbb{Z}h$

③ $h^2 = \deg E$

④ $[K_X] = -2h + (\deg E + 2g_C - 2) f \in H^2(X, \mathbb{Z})$

In particular, $K_X^2 = 8(1-g_C)$

pf: ① $\mathrm{Pic}(X) = \mathbb{Z}h + \langle D \in \mathrm{Pic}(X) \mid D.f = 0 \rangle$

Claim: $\langle D \in \mathrm{Pic}(X) \mid D.f = 0 \rangle = p^*\mathrm{Pic}(C)$

Set $D_n = \underline{D} + nF \qquad D_n.f = 0 \qquad D_n.K_X = D.K_X - 2n$

Riemann-Roch $\quad \chi(D_n) = \chi(\mathcal{O}_X) + \frac{1}{2} D_n \cdot (D_n - K_X) = n + \text{const.} \underset{n \gg 0}{>} 0$

$\| \quad$ Serre duality

$h^0(D_n) - h^1(D_n) + \underline{h^0(K_X - D_n)}$ $\quad$ Otherwise $\exists\, E \sim K_X - D_n$, $E \geq 0$

$\wedge\!\backslash \qquad\qquad\qquad \overset{\|}{0} \quad /$

$h^0(D_n)$ $\quad E.F \geq 0 \quad \because |F|$ no base point

$\quad \| $

$(K_X - D).F = -2n < 0$

$\because \exists\, E' \in |D_n|$ effective, $E'.f = 0 \Rightarrow E'$ union of fibres

② $H^1(X, \mathcal{O}_X^*) \twoheadrightarrow \underline{H^2(X, \mathbb{Z})} \longrightarrow H^2(X, \mathcal{O}_X) = 0$

$$\begin{array}{c} H^1(X,\mathcal{O}_X^*) \\ \cong \\ \mathrm{Pic}(X) \\ \cong \\ \boxed{p^*\mathrm{Pic}(C)} \oplus \mathbb{Z}h \end{array} \qquad \mathbb{Z}f$$

$\therefore \quad H^2(X, \mathbb{Z}) \simeq \mathbb{Z}\langle f, h \rangle$

intersection w/ f or h $\longrightarrow$ $\cong$ $\mathbb{Z}f \oplus \mathbb{Z}h$

③ $0 \longrightarrow N \longrightarrow p^*E \longrightarrow \mathcal{O}(1) \longrightarrow 0$

$$\wedge^2 p^*E \cong N \otimes \mathcal{O}(1)$$

$$\therefore \quad c_1(N) + h = (\deg E) f$$

$$\underline{c_1(N).h} + h^2 = \deg E$$

$$\|$$

$$c_2(p^*E) = p^*c_2(E) = 0$$

④ Assume that $K_X = ah + bf$

$$K_X . h = ah^2 + bf.h = a(\deg E) + b$$

$\|$ adjunction formula

$$\deg K_C - h^2 = 2g_c - 2 - \deg E$$

$$K_X . f = ah.f + bf^2 = a$$

$\|$ adjunction formula

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Birational invariant of ruled surfaces

Proposition 3. $q(X) = g_C$, $p_g(X) = 0$, $P_n(X) = 0$, $\forall n \in \mathbb{N}$

pf: $q(X)$, $P_g(X)$, $P_n(X)$ are birational invariants.

We may assume that $X \cong C \times \mathbb{P}^1$.

$$q(X) := h^1(X, \mathcal{O}_X) \underset{\text{Hodge decomposition}}{=} \frac{1}{2} b_1(C \times \mathbb{P}^1) \underset{\text{Kunneth formula}}{=} \frac{1}{2}(2g_C \times 1 + 1 \times 0) = g_C$$

$$P_g(X) := h^2(X, \mathcal{O}_X) \underset{\text{Serre duality}}{=} h^0(X, K_X) \underset{X = C \times \mathbb{P}^1}{=} h^0(C \times \mathbb{P}^1, K_C \boxtimes K_{\mathbb{P}^1})$$

$$= h^0(C, K_C) \otimes \underbrace{h^0(\mathbb{P}^1, K_{\mathbb{P}^1})}_{=0}$$

Similar for $P_n(X)$.

Proof of Theorem 2.

Step 1. $\mathbb{P}_C(E)$ is minimal i.e contains no (-1)-curve

Otherwise,
$$\begin{tikzcd} \mathbb{P}_C(E) \arrow[r, "p"] & C \\ \cup & \\ \text{exceptional curve} \arrow[r] & \text{pt or } C \end{tikzcd}$$

- Then exceptional curve is a fibre $f^2 = 0 \neq -1$
- C is NOT rational, no $\mathbb{P}^1 \to C$

Step 2.
$$\begin{tikzcd} X' \arrow[d, "\varepsilon_n"'] \arrow[ddr, "f"] & \\ \vdots \arrow[d, "\varepsilon_1"'] & \\ X \arrow[r, dashed] \arrow[dr, dashed] & C \\ & \mathbb{P}_C(E) \arrow[u, "p"'] \end{tikzcd}$$

ε_i: blow up of a point

X': minimal resolution of X to eliminate the discrepancy

Exceptional curve of ε_n is contracted by f $\because C \not\cong \mathbb{P}^1$

Contradict to minimality of X'.

Thus, $X' \cong X$ and
$$\begin{tikzcd} X \arrow[r] \arrow[dr, dashed] & C \\ & \mathbb{P}_C(E) \arrow[u, "p"'] \end{tikzcd}$$

Step 3: The theorem then follows from the lemma.

Lemma 1: If • X minimal surface, C smooth curve
• $X \xrightarrow{p} C$ generic fibres are $\mathbb{P}^1$

$\Rightarrow$ $X \to C$ is geometrically ruled.

pf: • (Zariski's lemma)
Fibres of p are irreducible.
C_i components of a fibre
then $(C_i . C_j)$ is negative definite

In particular, $C_i^2 < 0$

$\underset{\wedge 0}{C_i^2} + C_i . K_X = \deg K_{C_i} \geq -2 \Rightarrow C_i . K_X \underline{\geq} -1$

= holds iff $C_i^2 = -1$
C_i rational

$\not\to$ X is minimal.

$\therefore C_i . K_X \geq 0 \Rightarrow -2 = K_X . F = K_X . \sum_i m_i C_i \geq 0$ $\not\to$

• F fibre of p, irreducible

$F^2 = 0$, $F . K_X = -2$

F can't be multiple & $F \cong \mathbb{P}^1$ by genus formula.

Proof of Zariski's lemma:

Let $F = \sum_i \underbrace{m_i C_i}_{G_i}$, $m_i > 0$, C_i irreducible component of F

then F connected, $F^2 = 0 \implies F.C_i \underset{\le}{=} 0$

Let $D = \sum_i \underbrace{r_i C_i}_{s_i G_i} \neq 0 \qquad s_i = \frac{r_i}{m_i}$

$$D^2 = \sum_i s_i^2 \underbrace{G_i^2} + 2 \sum_{i \neq j} s_i s_j \, G_i . G_j$$

$$= \sum_i s_i^2 \underbrace{G_i . \left(F - \sum_{j \neq i} G_j\right)} + 2 \sum_{i \neq j} s_i s_j \, G_i . G_j$$

$$= \sum_i s_i^2 \underbrace{G_i . F}_{= 0 \ \ge} - \sum_{i \neq j} (s_i - s_j)^2 \underbrace{G_i . G_j}_{\ge 0} \le 0$$

$D^2 = 0 \iff s_i = s_j$ or $D = mF$ for some m.