

Ruled Surfaces

Note Title

5/20/2020

Definition: A surface X is ruled if it is birational to $C \times \mathbb{P}^1$, C : smooth curve.

ex. ① $C \times \mathbb{P}^1$

② $X = \mathbb{P}(E)$, E rank 2 vector bundle.
 \downarrow
 C

temporary auxiliary

Definition: A surface X is geometrically ruled if

$\exists X \xrightarrow{p} C$ smooth morphism, $p^{-1}(c) \cong \mathbb{P}^1, \forall c \in C$
Smooth curve

Theorem 1. If $X \xrightarrow{p} C$ is geometrically ruled, then

Enrique - Noether

$\exists U \subseteq C$ s.t. $p^{-1}(U) \cong U \times \mathbb{P}^1$
Zariski open

In particular, geometrically ruled \Rightarrow ruled

pf: Step 1. $H^2(X, \mathcal{O}_X) = 0$

Otherwise, $0 \neq H^2(X, \mathcal{O}_X) \cong H^0(X, K_X)$
serre duality

ie. \exists effective anti-canonical divisor K_X

F : generic fibre

$$\mathcal{O}_{\mathbb{P}^1}(-2) \cong K_F \cong F + K_X|_F$$

deg ↙
-2

adjunction formula

deg ↘

$F \cdot F + K_X \cdot F$
 \parallel
 0 \parallel
 0 0 K_X effective
 $\mathcal{O}(F)$ no base point

Step 2. \exists divisor S s.t. $S \cdot F = 1$. Later S can be realized as a section.

$$H^1(X, \mathcal{O}_X^*) \xrightarrow{G} H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathcal{O}_X) = 0$$

It suffices to prove that $\exists s \in H^2(X, \mathbb{Z})$ s.t. $s \cdot f = 1$ $[F]$
 \parallel
 f

Poincaré duality: $H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \rightarrow \mathbb{Z}$ non-degenerate

$$\Rightarrow 0 \rightarrow \text{Tor} \rightarrow H^2(X, \mathbb{Z}) \rightarrow \text{Hom}(H^2(X, \mathbb{Z}), \mathbb{Z})$$

$$K_X \cdot f = -2 \quad \overset{u}{f} \longmapsto (a \mapsto a \cdot f)$$

$\therefore f \notin \text{Tor}$

$$(a \cdot f) \triangleleft \mathbb{Z} \text{ ideal}$$

$$\parallel$$

$$d\mathbb{Z} \quad d=1 \text{ or } 2 \text{ in this case}$$

Poincaré duality $\Rightarrow \exists f' \in H^2(X, \mathbb{Z})$ s.t. $a \cdot f' = \frac{1}{d} a \cdot f, \forall a \in H^2(X, \mathbb{Z})$

$\therefore f = df'$ modulo torsion

Observation: $a^2 + a \cdot K_X \in (2) \triangleleft \mathbb{Z}, \forall a \in H^2(X, \mathbb{Z})$

Proposition 1: $X \xrightarrow{p} C$ geometrically ruled surface
 then $\exists E$ rank 2 vector bundle on C

s.t. $X \xrightarrow{\cong} \mathbb{P}(E)$ always projective

In particular, the quotient sub-bundles of E
 is equivalent to sections of p

pf: Notice that in the Step 3 of
 the proof of Theorem 1, one
 can run the argument again w/ $F = F_C$
 $\rightsquigarrow X \rightarrow C$ is locally trivial.

& classified by $H^1(C, \text{PGL}(2, \mathcal{O}_C))$

$$1 \rightarrow \mathbb{C}^* \rightarrow \text{GL}(2, \mathbb{C}) \rightarrow \text{PGL}(2, \mathbb{C}) \rightarrow 1$$

global
version

$$1 \rightarrow \mathcal{O}_C^* \rightarrow \text{GL}(2, \mathcal{O}_C) \rightarrow \text{PGL}(2, \mathcal{O}_C) \rightarrow 1$$

$$H^1(C, \mathcal{O}_C^*) \rightarrow H^1(C, \mathrm{GL}(2, \mathcal{O}_C)) \rightarrow H^1(C, \mathrm{PGL}(2, \mathcal{O}_C)) \rightarrow H^2(C, \mathcal{O}_C^*) = 0$$

Grothendieck



Thus all geometrically ruled surfaces are of the form $\mathbb{P}_C(E)$

$$\mathbb{P}_C(E) \cong \mathbb{P}_C(E') \iff E \cong E' \otimes \mathcal{L}, \quad \mathcal{L} \in H^1(C, \mathcal{O}_C^*)$$

ex. If $C \cong \mathbb{P}^1$, then $E \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$

Grothendieck

$$\mathbb{P}_C(E) \cong \mathbb{P}_C(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)) \cong \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a-b)) = \mathbb{F}_{|a-b|}$$

All geometric ruled surfaces over \mathbb{P}^1 are Hirzebruch surfaces $\mathbb{F}_n, n \in \mathbb{Z}$

Thus, we want to deviate slightly to understand rank 2 bundles on curves.

Lemma 1: $E =$ rank 2 vector bundle on C

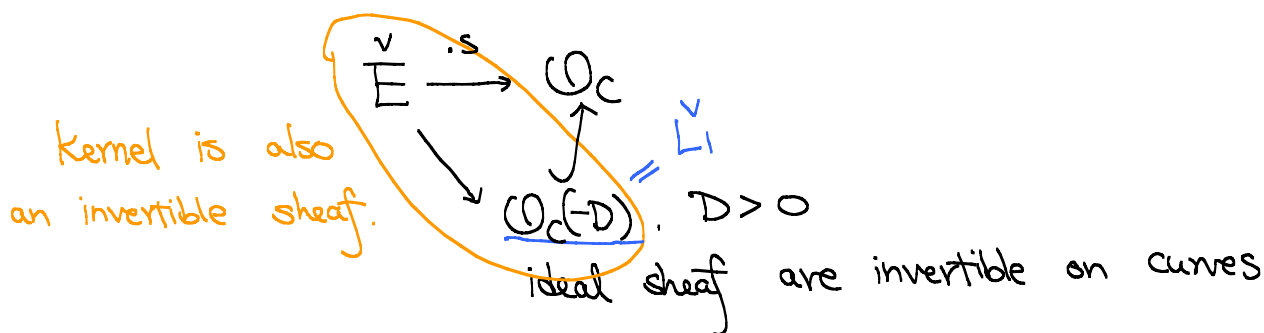
① \exists line bundles L_1, L_2

$$\text{s.t.} \quad 0 \rightarrow L_1 \rightarrow E \rightarrow L_2 \rightarrow 0$$

② If $h^0(E) \geq 1$, then can take $L_1 \cong \mathcal{O}_C(D), D \geq 0$.

(③ If $h^0(E) \geq 2, \deg E > 0$, can assume $D > 0$)

pf: Notice that every proper curve is projective.
 By tensoring high power of ample line bundle,
 we may assume that $\exists s \in H^0(E)$.
 by the theorem of Serre



Taking dual implies ① & ②

③ Notice that $D = \{s=0\}$ above

$h^0(E) \geq 2$ i.e. $\exists s_1, s_2 \in H^0(E)$

$\Rightarrow s_1, s_2 \in H^0(\tilde{\lambda}E)$ has a zero p
 $\deg \tilde{\lambda}E > 0$

i.e. $\lambda_1 s_1 + \lambda_2 s_2(p) = 0$ for some λ_1, λ_2
 $H^0(E)$

In particular, any rank 2 bundle E on C
 is an extension of line bundles of L_1, L_2
 parametrized by $\text{Ext}^1(L_1, L_2)$
 $\text{Ext}^1(\mathcal{O}_X, L_1^{-1} \otimes L_2) = H^1(L_1^{-1} \otimes L_2)$

Indeed $0 \rightarrow L_1 \otimes L_2^{-1} \rightarrow E \otimes L_2^{-1} \rightarrow \mathcal{O}_C \rightarrow 0$

$$E = L_1 \otimes L_2 \iff \begin{array}{ccccc} H^0(E \otimes L_2^{-1}) & \rightarrow & H^0(\mathcal{O}_C) & \xrightarrow{\alpha} & H^1(L_1 \otimes L_2^{-1}) \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ S & \xrightarrow{\quad} & 1 & \xrightarrow{\quad} & \alpha(1) = 0 \end{array}$$

i.e. $\alpha(1) \in H^1(L_1 \otimes L_2^{-1})$ is the obstruction for splitting.

ex. (elliptic curve) E indecomposable

$E \cong E' \otimes L$, $L \in \text{Pic}(C)$
 twist s.t. $\deg E' = 0, 1$ $E' =$ non-trivial extension of \mathcal{O}_C by \mathcal{O}_C
 or $= \mathcal{O}_C(p)$ by \mathcal{O}_C

ex. (genus $g \geq 2$ curve)

$$\begin{array}{ccc} E_s \rightarrow \mathcal{E} & E_s \not\cong E_{s'} \text{ if } s \neq s' & \\ \downarrow & \nearrow \lambda E_s \text{ constant} & \\ s \in S & E_s \text{ indecomposable} & \end{array}$$

$$\dim S = 3g - 3$$

Theorem 2. $C =$ smooth curve, not \mathbb{P}^1
 Then minimal models of $C \times \mathbb{P}^1$ is $\mathbb{P}_C(V)$
 for some V , $\text{rk}(V) = 2$

$$\textcircled{2} \quad H^1(X, \mathcal{O}_X^*) \rightarrow H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathcal{O}_X) = 0$$

$$\begin{array}{ccc} \begin{array}{c} \cong \\ \mathbb{P}^1 \\ \cong \\ \mathbb{P}^1 \end{array} & \begin{array}{c} \nearrow \\ \mathbb{Z}f \end{array} & \therefore H^2(X, \mathbb{Z}) \cong \mathbb{Z}\langle f, h \rangle \\ \boxed{\mathbb{P}^1 \mathbb{P}^1(\mathbb{C}) \oplus \mathbb{Z}h} & & \begin{array}{c} \text{intersection w/} \\ f \text{ or } h \end{array} \xrightarrow{\cong} \mathbb{Z}f \oplus \mathbb{Z}h \end{array}$$

$$\textcircled{3} \quad 0 \rightarrow N \rightarrow p^*E \rightarrow \mathcal{O}(1) \rightarrow 0$$

$$\wedge^2 p^*E \cong N \otimes \mathcal{O}(1)$$

$$\therefore c_1(N) + h = (\deg E) f$$

$$\frac{c_1(N) \cdot h + h^2}{=} = \deg E$$

$$c_2(p^*E) = p^*c_2(E) = 0$$

$$\textcircled{4} \quad \text{Assume that } K_X = ah + bf$$

$$K_X \cdot h = ah^2 + bf \cdot h = a(\deg E) + b$$

$$\begin{array}{c} \parallel \text{adjunction} \\ \text{formula} \end{array} \quad \deg K_X \cdot h^2 = 2g_C - 2 - \deg E$$

$$K_X \cdot f = ah \cdot f + bf^2 = a$$

$$\begin{array}{c} \parallel \text{adjunction} \\ \text{formula} \\ -2 \end{array}$$

Proof of Theorem 2.

Step 1. $\mathbb{P}_C(E)$ is minimal i.e contains no (-1) -curve

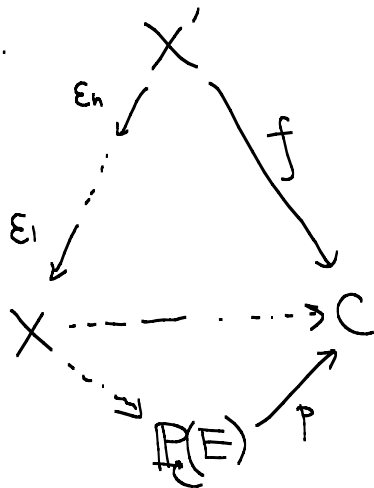
Otherwise, $\mathbb{P}_C(E) \xrightarrow{p} C$

exceptional curve \rightarrow pt or

• Then exceptional curve is a fibre
 $f^2 = 0 \neq -1$

• C is NOT rational
 no $\mathbb{P}^1 \rightarrow C$

Step 2.



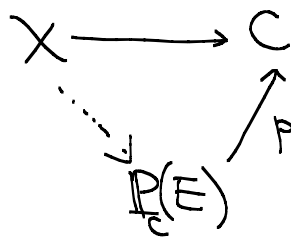
ε_i : blow up of a point

X' : minimal resolution of X
 to eliminate the discrepancy

Exceptional curve of ε_n
 is contracted by $f \because C \neq \mathbb{P}^1$

Contradict to minimality of X' .

Thus, $X' \cong X$ and



Step 3: The theorem then follows from the lemma.

Lemma 1: If \bullet X minimal surface, C smooth curve
 \bullet $X \xrightarrow{p} C$ generic fibres are \mathbb{P}^1
 $\Rightarrow X \rightarrow C$ is geometrically ruled.

pf: \bullet (Zariski's lemma)
 Fibres of p are irreducible.
 C_i components of a fibre
 then (C_i, C_j) is negative definite

In particular, $C_i^2 < 0$

$$C_i^2 + C_i \cdot K_X = \deg K_{C_i} \geq -2 \Rightarrow C_i \cdot K_X \geq -1$$

= holds iff $C_i^2 = -1$
 C_i rational
 $\Rightarrow X$ is minimal.

$$\therefore C_i \cdot K_X \geq 0 \Rightarrow -2 = K_X \cdot F = K_X \cdot \sum_i m_i C_i \geq 0 \quad \rightarrow$$

\bullet F fibre of p , irreducible

$$F^2 = 0, F \cdot K_X = -2$$

F can't be multiple & $F \cong \mathbb{P}^1$ by genus formula.

Proof of Zariski's lemma:

Let $F = \sum_i \underbrace{m_i}_{G_i} C_i$, $m_i > 0$, C_i irreducible component of F

then F connected, $F^2 = 0 \Rightarrow F \cdot C_i \leq 0$

$$\text{Let } D = \sum_i \underbrace{r_i C_i}_{s_i G_i} \neq 0 \quad s_i = \frac{r_i}{m_i}$$

$$D^2 = \sum_i s_i^2 \underbrace{G_i^2} + 2 \sum_{i \neq j} s_i s_j G_i \cdot G_j$$

$$= \sum_i s_i^2 \underbrace{G_i (F - \sum_{j \neq i} G_j)} + 2 \sum_{i \neq j} s_i s_j G_i \cdot G_j$$

$$= \sum_i s_i^2 \underbrace{G_i \cdot F}_{=0} - \sum_{i \neq j} (s_i - s_j)^2 \underbrace{G_i \cdot G_j}_{=0} \leq 0$$

$$D^2 = 0 \Leftrightarrow s_i = s_j \quad \text{or} \quad D = mF \quad \text{for some } m.$$